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METHODS OF CALCULATING THE ROOTS OF THE DISPERSION EQUATION FOR FREE OSCILLATIONS OF A GAS BUBBLE IN A LIQUID

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The problem of free radial oscillations of gas bubbles in a liquid is considered. The structure of the roots of the dispersion equation in the presence of heat transfer between the phases is studied in detail. It is shown that this equation has two complex-conjugate roots and an infinite number of real roots; all of the roots lie in the left complex half-plane, providing damping of radial oscillations. Approximate expressions for these roots are obtained.

Key words: *dispersion equation, gas bubbles in liquid, free oscillations.*

A gas bubble freely oscillating in an ideal incompressible liquid is considered assuming spherical symmetry of the process and uniform pressure in the bubbles. The latter is the case when the sound wavelength in the gas is much larger than the bubble size.

Under the adopted assumptions, the heat-transfer and continuity equations for the gas phase in the spherical Eulerian coordinates r and t are written as

$$\rho c_p\Big(\frac{\partial T}{\partial t} + w\,\frac{\partial T}{\partial r}\Big) = \frac{1}{r^2}\,\frac{\partial}{\partial r}\Big(\lambda r^2\,\frac{\partial T}{\partial r}\Big) + \frac{dp}{dt},$$
$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r^2\rho w) = 0, \qquad (1)$$

where ρ is the density, T is the temperature, w is the radial velocity, p is the pressure, c_p is the heat capacity at constant pressure, and λ is the thermal conductivity.

As shown in [1], the boundary condition for the bubble surface temperature can be written as

$$r = a(t): \qquad T = T_0 = \text{const}, \qquad (2)$$

where a is the bubble radius and T_0 is the liquid temperature.

The boundary condition at the center of the bubble is given by

$$r = 0: \qquad \frac{\partial T}{\partial r} = 0. \qquad (3)$$

The dynamics of radial oscillation of the bubble is described by the Rayleigh equation [2]

$$a\,\frac{d^2a}{dt^2} + \frac{3}{2}\Big(\frac{da}{dt}\Big)^2 = \frac{p - p_\infty - 2\sigma/a}{\rho_e}, \qquad (4)$$

where p_∞ is the pressure in the liquid far from the bubble, σ is the surface tension coefficient, and ρ_e is the liquid density.

The equation of state for the gas is given by

$$p = \rho RT, \qquad (5)$$

where R is the gas constant.

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Using the condition of pressure uniformity in the bubble, from (1) and (5) we obtain the relation

$$\frac{dp}{dt} = \frac{3(\gamma - 1)}{a} \lambda \Big(\frac{\partial T}{\partial r}\Big)_a - \frac{3\gamma p}{a} \frac{da}{dt}, \tag{6}$$

where γ is the adiabatic exponent of the gas.

In the case of small oscillations, the bubble radius is given by the real part of the expression

$$a = a_0[1 + \alpha \exp{(ht)}],$$

where the subscript 0 corresponds to the parameters in the unperturbed state, $|\alpha| \ll 1$ is the complex number, and $\omega = \operatorname{Im} h$ is the frequency of the oscillations.

Let us linearize the basic equations (1)–(6). It is assumed that the small deviations of the pressure and temperature from the corresponding values in the state of equilibrium have the form [1–3]

$$p = p_0[1 + P \exp{(ht)}], \qquad T = T_0[1 + \theta(r) \exp{(ht)}].$$

After the linearization and transformation to nondimensional quantities, system (1)–(6) can be written as

$$H\theta = \nabla^2\theta + (1 - 1/\gamma)HP; \tag{7}$$

$$P = \frac{\alpha H^2}{N} - S\alpha, \qquad HP = 3\gamma\Big(\frac{\partial \theta}{\partial \xi}\Big|_{\xi=1} - \alpha H\Big), \tag{8}$$

where

$$\xi = \frac{r}{a_0}, \quad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{2}{\xi}\frac{\partial}{\partial \xi}, \quad S = \frac{2\sigma}{a_0 p_0}, \quad N = \frac{p_0 a_0^2}{\rho_e D^2}, \quad D = \frac{\lambda}{\rho c_p}, \quad H = \frac{h a_0^2}{D}.$$

The solution of Eq. (7) with boundary conditions (2) and (3) is given by

$$\theta = \Big(1 - \frac{1}{\gamma}\Big)P\Big(1 - \frac{\sinh{(H^{1/2}\xi)}}{\xi \sinh{(H^{1/2})}}\Big). \tag{9}$$

The existence condition for a nontrivial solution of the linear equations (7)–(9) leads to the characteristic equation for H — the transcendental equation [2]

$$f(H) = f_0(H) + f_1(H)\operatorname{Kh}(H) = 0; \tag{10}$$

$$\operatorname{Kh}(H) = H^{1/2} \coth{(H^{1/2})} - 1, \tag{11}$$

where $f_0(H)$ and $f_1(H)$ are polynomials of H, the degree of the second polynomial being lower than the degree of the first polynomial.

In the case of large bubbles, where capillary effects can be ignored, we have $f_0(H) = H^2 + A^2$, $f_1(H) = \varepsilon H$, $A = \sqrt{3\gamma N}$, and $\varepsilon = 3(\gamma - 1)$.

The last term -1 in the expression for $\operatorname{Kh}(H)$ (11) can be combined with the first polynomial in (10). However, in mechanical problems, the coefficient at $\operatorname{Kh}(H)$ has a physical meaning, and this combination therefore makes no sense.

We note that $\operatorname{Kh}(H)$ is a meromorphic function in the entire complex plane ζ (no branching at the points $\zeta = 0$ and $\zeta = \infty$); therefore, the function $\operatorname{Kh}(H)$ does not depend on the choice of the sign ahead of the square root. Therefore, we can set $\sqrt{H} = x + iy$ for $x > 0$ or $x = 0$ and $y \geqslant 0$. We prove the following lemma.

Lemma 1. *The function* $\coth{(x + iy)}$ *and its modulus can be represented as*

$$\coth{(x + iy)} = \frac{\sinh{(2x)} - i\sin{(2y)}}{\cosh{(2x)} - \cos{(2y)}},$$

$$|\coth{(x + iy)}| = \sqrt{\frac{\cosh{(2x)} + \cos{(2y)}}{\cosh{(2x)} - \cos{(2y)}}} = \sqrt{\frac{\sinh^2 x + \cos^2 y}{\sinh^2 x + \sin^2 y}}.$$

Proof. The function $\coth{(x + iy)}$ can be written as

$$\coth{(x + iy)} = \frac{\exp{(x + iy)} + \exp{(-x - iy)}}{\exp{(x + iy)} - \exp{(-x - iy)}} = \frac{\cosh x \cos y + i \sinh x \sin y}{\sinh x \cos y + i \cosh x \sin y}.$$

Multiplying the numerator and denominator by a number which is complex conjugate to the denominator, we obtain

$$\coth\,(x+iy)=\frac{\cosh\,x\sinh\,x(\cos^2 y+\sin^2 y)+i\sin y\cos y(\sinh^2 x-\cosh^2 x)}{\sinh^2 x\cos^2 y+\cosh^2 x\sin^2 y}.$$

Transforming to binary arguments (by multiplying the numerator and denominator by 2), we obtain

$$\coth\,(x+iy)=\frac{\sinh\,(2x)-i\sin(2y)}{(\cosh\,(2x)-1)\cos^2 y+(\cosh\,(2x)+1)\sin^2 y}=\frac{\sinh\,(2x)-i\sin(2y)}{\cosh\,(2x)-\cos(2y)}.$$

Calculation of the modulus of this expression yields

$$|\coth\,(x+iy)|=\sqrt{\frac{\sinh^2(2x)+\sin^2(2y)}{(\cosh\,(2x)-\cos(2y))^2}}.$$

Taking into account that

$$\sinh^2(2x)+\sin^2(2y)=\cosh^2(2x)-1+\sin^2(2y)=\cosh^2(2x)-\cos^2(2y),$$

and decomposing the terms of this expression into multipliers, we obtain

$$|\coth\,(x+iy)|=\sqrt{\frac{\cosh\,(2x)+\cos(2y)}{\cosh\,(2x)-\cos(2y)}}.$$

Transformation from the binary arguments to ordinary arguments leads to the required representation for $|\coth\,(x+iy)|$. The representation

$$\coth\,(x+iy)=\frac{\sinh\,(2x)-i\sin(2y)}{\cosh\,(2x)-\cos(2y)}$$

implies that the poles of the function $\mathrm{Kh}\,(H)$ are the points

$$x=0,\quad y=\pi n,\qquad H=(iy)^2=-(\pi n)^2,\qquad n=1,2,\ldots,$$

and the zeros are the solutions of the equation $\tan y=y$ and $H=-y^2$.

The poles of the function $\mathrm{Kh}\,(H)$ are also poles for the characteristic function $f(H)$, and zeros of the form $H_n=-y_n^2$, where $n\pi<y_n<(n+1)\pi$, are also zeros of the function $f(H)$ because the function $f(-y^2)$ takes real values, and in each of the specified intervals, it varies from $-\infty$ to $+\infty$. These zeros [calculated as the real roots of the equation $f_0(-y^2)+f_1(-y^2)(y\cot\,(y)-1)=0$] define rapidly decreasing solutions without oscillations (H is a negative real number). The last equation can be written as

$$\tan y=\frac{yf_1(-y^2)}{f_0(-y^2)-f_1(-y^2)},\tag{12}$$

from which it follows that, in the interval $n\pi<y_n<(n+1)\pi$, the characteristic equation has an infinite number of real roots. Figure 1 shows the qualitative shape of the curves described by the left and right sides of Eq. (12) for the case $f_0(H)=H^2+A^2$, $f_1(H)=\varepsilon H$, $A=\sqrt{3\gamma N}$, and $\varepsilon=3(\gamma-1)$ in the problem of oscillations of a large bubble taking into account temperature nonuniformity inside the bubble.

The question arises of the existence of other zeros (zeros can also appear in the negative part of the straight line). Of special interest are zeros with a positive real part (instability). We consider the region bounded by the straight line $y\leqslant\pi(n+1/4)$ for $\sqrt{H}=x+iy$. In the plane H, this region is bounded by the parabola (Fig. 2):

$$\mathrm{Re}\,H\geqslant\Big(\frac{\mathrm{Im}\,H}{2\pi(n+1/4)}\Big)^2-\pi^2(n+1/4)^2.$$

Because, according to Lemma 1, on the boundary of the region and at infinity, $|\coth\,(x+iy)|=1$, it follows that for large H, the inequality $|f_0(H)|>|f_1(H)|$ holds. Consequently, along the boundary of the region considered, the argument of the function $f(H)$ varies in the same manner as the argument of the function $f_0(H)$. From the argument principle [4], it follows that for large n, the difference in number between zeros and poles in this region is equal to $N-P=\deg\,(f_0)$. In particular, the above characteristic function has not only real zeros but also a pair of complex-conjugate zeros. This property holds for any regular characteristic function (all complex zeros are included in the pairs of complex-conjugate numbers).

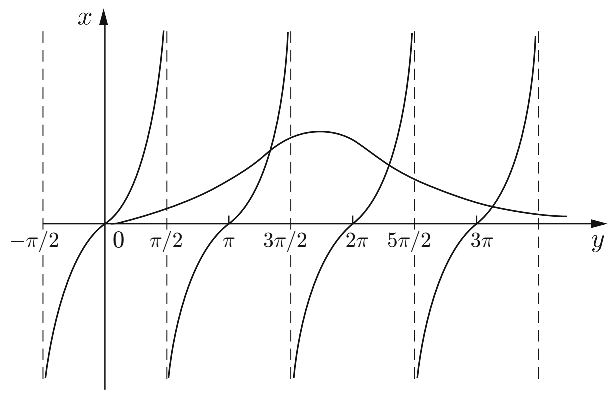

Fig. 1

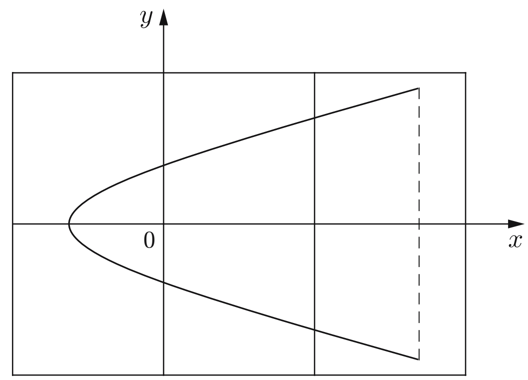

Fig. 2

Fig. 1. Graphical method of determining the roots of Eq. (12).

Fig. 2. Contour of the region in which the argument principle is applied.

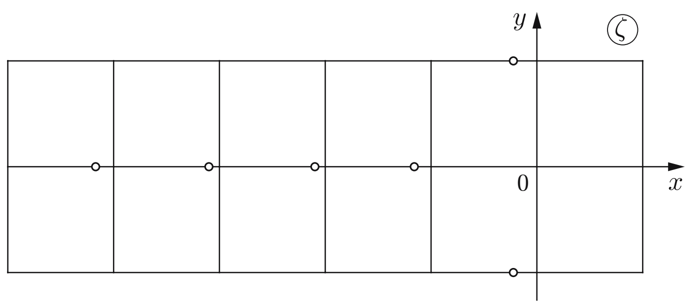

Fig. 3. Positions of the roots of the characteristic equation (10) in the plane ζ.

In the example considered, the function $f_1(H)$ has a small multiplier. In this case, in the absence of multiple roots of the function $f_0(H) = 0$, the search of nontrivial zeros of the characteristic function reduces to the following iterative procedure. Let H_0 be some zero of $f_0(H)$; then, $H_{k+1} = f_0^{-1}(-f_1(H_k)\,\mathrm{Kh}\,(H_k))$ [f_0^{-1} is a local inverse function near the corresponding zero of the function $f_0(H)$].

We calculate the roots of the characteristic function $H^2 + A^2 + \varepsilon H(H^{1/2}\coth\,(H^{1/2}) - 1) = 0$ for the first approximation for ε:

$$H \approx iA\sqrt{1 + (\varepsilon i/A)\,\mathrm{Kh}\,(iA)} \approx iA - (\varepsilon/2)\,\mathrm{Kh}\,(iA). \tag{13}$$

Introducing $a = \sqrt{2A}$, taking into account that $H^{1/2} = a(1+i)/2$, and using Lemma 1, we obtain the expression

$$\mathrm{Kh}\,(iA) = \frac{a(1+i)}{2}\,\frac{\sinh\,a - i\sin a}{\cosh\,a - \cos a} - 1 = \frac{a(\sinh\,a + \sin a)}{2(\cosh\,a - \cos a)} - 1 + i\,\frac{a(\sinh\,a - \sin a)}{2(\cosh\,a - \cos a)}.$$

Substitution of this expression into formula (13) yields approximate values for two conjugate roots:

$$H \approx \pm iA\Big(1 - \frac{\varepsilon}{2}\,\frac{\sinh\,a - \sin a}{\cosh\,a - \cos a}\Big) - \frac{\varepsilon}{2}\Big(\frac{(a/2)(\sinh\,a + \sin a) - \cosh\,a + \cos a}{\cosh\,a - \cos a}\Big).$$

It should be noted that the expression in square brackets is always positive; therefore the real part of the roots is negative. In this case, all the real roots are larger in absolute values than the real parts of the complex roots (Fig. 3).

It is easy to show that

$$0 < \frac{\sinh\,a - \sin a}{2a(\cosh\,a - \cos a)} < \frac{1}{6}.$$

From these inequalities, it follows that, as ε increases from zero, the frequency of the oscillations decreases only slightly. By differentiation, it is easy to show that, in the last fraction, both the denominator and numerator are positive. This implies that the corresponding solutions are more stable. In real situations, however, these solutions decrease (oscillating) much more slowly than even the most slowly decaying (without oscillation) solution (the value of ε is small). Previously, the argument principle was used in [5] to study the structure of the roots of the dispersion equation. A review of papers on this subject is given in [6].

Thus, it is proved that all roots of the characteristic equation, except for two complex-conjugate roots with a negative real part, are negative real numbers and are larger in absolute value than the real parts of the complex roots, i.e., they decay rapidly and do not make a significant contribution to the general solution. It should be noted that roots with a positive real part that cause instability are absent. This is evidence for the correctness of the results of papers [1, 3], in which characteristic transcendental equations were solved numerically and only complex-conjugate roots were found and were then used to calculate the damping decrement.

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